Formelsammlung Elektrotechnik

von Sascha Spors

V1.3 / 01.10.96

Mathematische Formeln

$$\underline{Z} = a + jb \quad \Rightarrow \arg(\underline{Z}) = \begin{cases} \arctan(\frac{b}{a}) & \text{für } a>0,\ b<>0 \\ \arctan(\frac{b}{a}) + \pi & \text{für } a<0,\ b>0 \\ \arctan(\frac{b}{a}) - \pi & \text{für } a<0,\ b<0 \end{cases}$$

$$\underline{Z} = \frac{a + jb}{c + jd} = \frac{N}{Z} \quad \Rightarrow \quad |\underline{Z}| = \sqrt{\frac{a^2 + b^2}{c^2 + d^2}} = \sqrt{\frac{|N|}{|Z|}} \qquad \arg(\underline{Z}) = \arg(N) - \arg(Z)$$

$$\underline{Z} = r\,e^{j\varphi} \quad \Rightarrow \quad \sqrt[n]{\underline{Z}} = \sqrt[n]{r}\ e^{j\psi} \ \text{mit } \psi = \frac{\varphi}{n} + \frac{2k\pi}{n},\ k = 0,1,\ldots,n-1 \qquad \underline{Z}\,\underline{Z}^* = |\underline{Z}|^2$$

$$\sin\varphi = \cos(\varphi - \frac{\pi}{2}) \qquad \cos\varphi = \sin(\varphi + \frac{\pi}{2}) \qquad \cos(\arctan\frac{a}{b}) = \frac{b}{\sqrt{a^2+b^2}}$$

$$\sin(\varphi \pm \pi) = -\sin\varphi \qquad \cos(\varphi \pm \pi) = -\cos\varphi \qquad \sin(\arctan\frac{a}{b}) = \frac{a}{\sqrt{a^2+b^2}}$$

$$e^{j\varphi} = \cos(\varphi) + j\sin(\varphi) \qquad \sin\varphi = \frac{1}{2j}(e^{j\varphi} - e^{-j\varphi}) \qquad \cos\varphi = \frac{1}{2}(e^{j\varphi} + e^{-j\varphi})$$

Physikalische Grundlagen

Maxwellsche Gleichungen

$$\varepsilon_0 \int_A\int \mathbf{E}\,d\mathbf{a} = Q \qquad \int_A\int \mathbf{B}\,d\mathbf{a} = 0$$

$$\int_K \mathbf{E}\,d\mathbf{r} = 0 \qquad \int_K \frac{\mathbf{B}}{\mu}\,d\mathbf{r} = \sum_\nu \pm i_\nu$$

Elektrisches Feld

$$\varphi(P) = -\int_{P_0}^{P} \mathbf{E}\,d\mathbf{r} \qquad u_{12} = \int_{P_1}^{P_2} \mathbf{E}\,d\mathbf{r} = \varphi(P_1) - \varphi(P_2) \qquad \Delta W = q\int_{P_1}^{P_2} \mathbf{E}\,d\mathbf{r}$$

Magnetisches Feld

$$\phi = \int_A\int \mathbf{B}\,d\mathbf{a} \qquad u_{12} = \frac{d\phi}{dt} \qquad \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Umrechnung Zeitbereich <-> Zeigerbereich

$$u(t) = \sqrt{2}\,U\cos(\omega t + \alpha) \quad \Leftrightarrow \quad \underline{U} = U\,e^{j\alpha}$$

$$i(t=0) = \frac{\sqrt{2}}{2}\left(\underline{I} + \underline{I}^*\right)$$

$$i(t) = \sqrt{2}\,I\cos(\omega t + \beta) \quad \Leftrightarrow \quad \underline{I} = I\,e^{j\beta}$$

$$u(t=0) = \frac{\sqrt{2}}{2}\left(\underline{U} + \underline{U}^*\right)$$

$$\arg(\underline{U},\underline{I}) := \arg(\underline{I}) - \arg(\underline{U}) = \beta - \alpha$$

Leistung

$$P = UI = RI^2 = \frac{U^2}{R}$$

Augenblicksleistung $p(t) = u(t)i(t) = UI\cos(\alpha - \beta) + UI\cos(2\omega t + \alpha + \beta)$

komplexe Leistung $\underline{P} = \underline{U}\underline{I}^* = |\underline{I}|^2\underline{Z} = |\underline{U}|^2\underline{Y}^* = P_w + jP_b$

Wirkleistung $P_w = \mathrm{Re}(\underline{P}) = |\underline{I}|^2\mathrm{Re}(\underline{Z}) = |\underline{U}|^2\mathrm{Re}(\underline{Y}) = \overline{p(t)}$

Blindleistung $P_b = \mathrm{Im}(\underline{P}) = |\underline{I}|^2\mathrm{Im}(\underline{Z}) = -|\underline{U}|^2\mathrm{Im}(\underline{Y})$

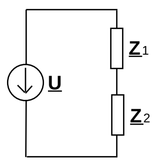
Scheinleistung $P_s = |\underline{P}|$

Arbeit $W = \int p(t)dt = \int u(t)i(t)\,dt$ Leistung $p = \frac{dW}{dt}$

Grundschaltungen

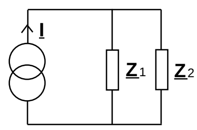
Spannungs-/Stromteilerschaltung

$$\underline{Z}_{Ges} = \underline{Z}_1 + \underline{Z}_2 \qquad \underline{Z}_{Ges} = \frac{\underline{Z}_1\underline{Z}_2}{\underline{Z}_1 + \underline{Z}_2}$$

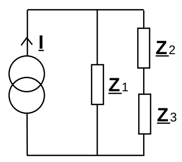


$$\underline{Y}_{Ges} = \frac{\underline{Y}_1\underline{Y}_2}{\underline{Y}_1 + \underline{Y}_2} \qquad \underline{Y}_{Ges} = \underline{Y}_1 + \underline{Y}_2$$

$$\underline{U}_1 = \frac{\underline{Z}_1}{\underline{Z}_1 + \underline{Z}_2}\,\underline{U} = \frac{\underline{Y}_2}{\underline{Y}_1 + \underline{Y}_2}\,\underline{U} \qquad \underline{I}_1 = \frac{\underline{Z}_2}{\underline{Z}_1 + \underline{Z}_2}\,\underline{I} = \frac{\underline{Y}_1}{\underline{Y}_1 + \underline{Y}_2}\,\underline{I}$$



erweiterte Spannungsteilerschaltung



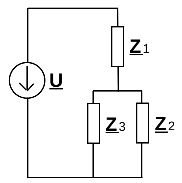
$$\underline{Z}_{Ges} = \frac{\underline{Z}_1(\underline{Z}_2+\underline{Z}_3)}{\underline{Z}_1+\underline{Z}_2+\underline{Z}_3} \qquad \underline{Y}_{Ges} = \frac{\underline{Y}_1\underline{Y}_2+\underline{Y}_1\underline{Y}_3+\underline{Y}_2\underline{Y}_3}{\underline{Y}_2+\underline{Y}_3}$$

$$\underline{I}_1 = \frac{\underline{Z}_2+\underline{Z}_3}{\underline{Z}_1+\underline{Z}_2+\underline{Z}_3}\underline{I} = \frac{\underline{Y}_1(\underline{Y}_2+\underline{Y}_3)}{\underline{Y}_1\underline{Y}_2+\underline{Y}_1\underline{Y}_3+\underline{Y}_2\underline{Y}_3}\underline{I}$$

$$\underline{I}_2 = \frac{\underline{Z}_1}{\underline{Z}_1+\underline{Z}_2+\underline{Z}_3}\underline{I} = \frac{\underline{Y}_2\underline{Y}_3}{\underline{Y}_1\underline{Y}_2+\underline{Y}_1\underline{Y}_3+\underline{Y}_2\underline{Y}_3}\underline{I}$$

$$\underline{U}_2 = \frac{\underline{Z}_1\underline{Z}_2}{\underline{Z}_1+\underline{Z}_2+\underline{Z}_3}\underline{I} = \frac{\underline{Y}_3}{\underline{Y}_1\underline{Y}_2+\underline{Y}_1\underline{Y}_3+\underline{Y}_2\underline{Y}_3}\underline{I}$$

erweiterte Stromteilerschaltung



$$\underline{Z}_{Ges} = \frac{\underline{Z}_1\underline{Z}_2+\underline{Z}_1\underline{Z}_3+\underline{Z}_2\underline{Z}_3}{\underline{Z}_2+\underline{Z}_3} \qquad \underline{Y}_{Ges} = \frac{\underline{Y}_1(\underline{Y}_2+\underline{Y}_3)}{\underline{Y}_1+\underline{Y}_2+\underline{Y}_3}$$

$$\underline{U}_1 = \frac{\underline{Z}_1(\underline{Z}_2+\underline{Z}_3)}{\underline{Z}_1\underline{Z}_2+\underline{Z}_1\underline{Z}_3+\underline{Z}_2\underline{Z}_3}\underline{U} = \frac{\underline{Y}_2+\underline{Y}_3}{\underline{Y}_1+\underline{Y}_2+\underline{Y}_3}\underline{U}$$

$$\underline{U}_2 = \frac{\underline{Z}_2\underline{Z}_3}{\underline{Z}_1\underline{Z}_2+\underline{Z}_1\underline{Z}_3+\underline{Z}_2\underline{Z}_3}\underline{U} = \frac{\underline{Y}_1}{\underline{Y}_1+\underline{Y}_2+\underline{Y}_3}\underline{U}$$

$$\underline{I}_2 = \frac{\underline{Z}_3}{\underline{Z}_1\underline{Z}_2+\underline{Z}_1\underline{Z}_3+\underline{Z}_2\underline{Z}_3}\underline{U} = \frac{\underline{Y}_1\underline{Y}_2}{\underline{Y}_1+\underline{Y}_2+\underline{Y}_3}\underline{U}$$

Schwingkreis

DG – Reihenschwingkreis: $\dfrac{d^2u_c}{d\tau^2} + \dfrac{1}{Q}\dfrac{du_c}{d\tau} + u_c = u_0(\tau/\omega_0)$

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \tau = \omega_0 t$$

Güte Q
$$\begin{cases} Q = \dfrac{\omega_0 L}{R} = \dfrac{1}{\omega_0 RC} = \dfrac{1}{R}\sqrt{\dfrac{L}{C}} & \text{im Reihenschwingkreis} \\[2ex] Q = \dfrac{R}{\omega_0 L} = \omega_0 RC = R\sqrt{\dfrac{C}{L}} & \text{im Parallelschwingkreis} \end{cases}$$

Bauteile

Widerstand

$\underline{U} = R\,\underline{I}$ $\quad p(t) = Ri^2(t) = \frac{u^2(t)}{R}$ $\quad [R] = 1\frac{V}{A} = 1\Omega$

Induktivität

Strom i(t) ist stetig

$u(t) = L\frac{di(t)}{dt}$ $\quad i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} u(\tau)d\tau$ $\quad W(t) = \frac{1}{2}L\left[\,i^2(t) - i^2(t_0)\right]$

$\underline{Z} = j\omega L$ $\quad [L] = 1\frac{Vs}{A} = 1H$ $\quad p(t) = L\,i(t)\,\frac{di(t)}{dt}$

Kapazität

Spannung u(t) ist stetig

$i(t) = C\frac{du(t)}{dt}$ $\quad u(t) = u(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau$ $\quad W(t) = \frac{1}{2}C\left[\,u^2(t) - u^2(t_0)\right]$

$\underline{Z} = \frac{1}{j\omega C} = -\frac{j}{\omega C}$ $\quad C = \frac{Q}{U}$ $\quad [C] = 1\frac{As}{V} = 1F$ $\quad p(t) = C\,u(t)\,\frac{du(t)}{dt}$

Übertrager

realer Übertrager

Strom $i_1(t)$ und $i_2(t)$ sind stetig

$L_1 L \geq M^2$ $\quad M <> 0$

2 Windungen w_1, w_2

$u_1(t) = L_1\frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt}$ $\quad \underline{U}_1 = j\omega L_1\underline{I}_1 + j\omega M\underline{I}_2$ $\quad \underline{I}_1 = \frac{L_2\underline{U}_1 - M\underline{U}_2}{j\omega(L_1L_2 - M^2)}$

$u_2(t) = M\frac{di_1(t)}{dt} + L_2\frac{di_2(t)}{dt}$ $\quad \underline{U}_2 = j\omega M\underline{I}_1 + j\omega L_2\underline{I}_2$ $\quad \underline{I}_2 = \frac{L_1\underline{U}_2 - M\underline{U}_1}{j\omega(L_1L_2 - M^2)}$

3 Windungen w_1, w_2, w_3

$u_1(t) = L_1\frac{di_1(t)}{dt} + M_{12}\frac{di_2(t)}{dt} + M_{13}\frac{di_3(t)}{dt}$ $\quad \underline{U}_1 = j\omega L_1\underline{I}_1 + j\omega M_{12}\underline{I}_2 + j\omega M_{13}\underline{I}_3$

$u_2(t) = M_{12}\frac{di_1(t)}{dt} + L_2\frac{di_2(t)}{dt} + M_{23}\frac{di_3(t)}{dt}$ $\quad \underline{U}_2 = j\omega M_{12}\underline{I}_1 + j\omega L_2\underline{I}_2 + j\omega M_{23}\underline{I}_3$

$u_3(t) = M_{13}\frac{di_1(t)}{dt} + M_{23}\frac{di_2(t)}{dt} + L_3\frac{di_3(t)}{dt}$ $\quad \underline{U}_3 = j\omega M_{13}\underline{I}_1 + j\omega M_{23}\underline{I}_2 + j\omega L_3\underline{I}_3$

festgekoppelter Übertrager

Magnetisierungsstrom $i_m(t)$ ist stetig

2 Windungen w_1, w_2

$i_m(t) = w_1 i_1(t) + w_2 i_2(t)$

$M = \pm\sqrt{L_1 L_2}$ $\qquad$ $ü = \frac{\underline{U}_1}{\underline{U}_2} = \frac{w_1}{w_2}$ $\qquad$ $ü^2 = \frac{L_1}{L_2}$

$L_1 = kw_1^2$ $\qquad$ $L_2 = kw_2^2$ $\qquad$ $M = kw_1 w_2$ $\qquad$ $\underline{\phi} = k(w_1 \underline{I}_1 + w_2 \underline{I}_2)$ $\qquad$ $\underline{U}_1 = j\omega w_1 \underline{\phi}$ $\qquad$ $\underline{U}_2 = j\omega w_2 \underline{\phi}$

n Windungen $w_1, w_2, \ldots, w_n$

$i_m(t) = w_1 i_1(t) + w_2 i_2(t) + \ldots\ldots + w_n i_n(t)$

$M_{\mu\nu} = \pm\sqrt{L_\mu L_\nu}$ $\qquad$ $L_\mu = kw_\mu^2$ $\qquad$ $M_{\mu\nu} = kw_\mu w_\nu$ $\qquad$ $\underline{\phi} = k(w_1 \underline{I}_1 + \ldots\ldots + w_n \underline{I}_n)$ $\qquad$ $\underline{U}_\mu = j\omega w_\mu \underline{\phi}$

idealer Übertrager

2 Windungen w_1, w_2

$i_m(t) = w_1 i_1(t) + w_2 i_2(t) = 0$

$\underline{I}_1 = -\frac{w_2}{w_1} \underline{I}_2$ $\qquad$ $\underline{U}_1 = w_1 \underline{U}_H$ $\qquad$ $\underline{U}_2 = w_2 \underline{U}_H$ $\qquad$ $w_1 \underline{I}_1 + w_2 \underline{I}_2 = 0$

n Windungen $w_1, w_2, \ldots, w_n$

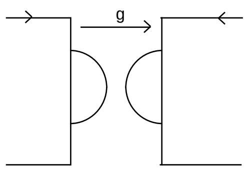
$i_m(t) = w_1 i_1(t) + w_2 i_2(t) + \ldots + w_n i_n(t) = 0$

$\underline{U}_\mu = w_\mu \underline{U}_H$ $\qquad$ $w_1 \underline{I}_1 + \ldots + w_{n_n} \underline{I}_{nn} = 0$ $\qquad$ $\underline{U}_H = j\omega k \underline{I}_M$

Gyrator

$\underline{I}_1 = g\underline{U}_2$ $\qquad$ $\underline{I}_2 = -g\underline{U}_1$

$\underline{U}_1 = -\frac{1}{g}\underline{I}_2$ $\qquad$ $\underline{U}_2 = \frac{1}{g}\underline{I}_1$



ideale Diode

Leitet: $u_D=0$, $i_D>0$
Sperrt: $i_D=0$, $u_D<0$

Zweitore

Impedanzmatrix $\underline{Z}$

$$\begin{bmatrix}\underline{U}_1\\ \underline{U}_2\end{bmatrix} = \begin{bmatrix}\underline{z}_{11} & \underline{z}_{12}\\ \underline{z}_{21} & \underline{z}_{22}\end{bmatrix} \begin{bmatrix}\underline{I}_1\\ \underline{I}_2\end{bmatrix}$$

$$\underline{z}_{11}=\frac{\underline{U}_1}{\underline{I}_1}\Big|_{\underline{I}_2=0} \qquad \underline{z}_{12}=\frac{\underline{U}_1}{\underline{I}_2}\Big|_{\underline{I}_1=0} \qquad \underline{z}_{21}=\frac{\underline{U}_2}{\underline{I}_1}\Big|_{\underline{I}_2=0} \qquad \underline{z}_{22}=\frac{\underline{U}_2}{\underline{I}_2}\Big|_{\underline{I}_1=0}$$

Admittanzmatrix $\underline{Y}$

$$\begin{bmatrix}\underline{I}_1\\ \underline{I}_2\end{bmatrix} = \begin{bmatrix}\underline{y}_{11} & \underline{y}_{12}\\ \underline{y}_{21} & \underline{y}_{22}\end{bmatrix} \begin{bmatrix}\underline{U}_1\\ \underline{U}_2\end{bmatrix}$$

$$\underline{y}_{11}=\frac{\underline{I}_1}{\underline{U}_1}\Big|_{\underline{U}_2=0} \qquad \underline{y}_{12}=\frac{\underline{I}_1}{\underline{U}_2}\Big|_{\underline{U}_1=0} \qquad \underline{y}_{21}=\frac{\underline{I}_2}{\underline{U}_1}\Big|_{\underline{U}_2=0} \qquad \underline{y}_{22}=\frac{\underline{I}_2}{\underline{U}_2}\Big|_{\underline{U}_1=0}$$

Kettenmatrix $\underline{A}$

$$\begin{bmatrix}\underline{U}_1\\ \underline{I}_1\end{bmatrix} = \begin{bmatrix}\underline{a}_{11} & \underline{a}_{12}\\ \underline{a}_{21} & \underline{a}_{22}\end{bmatrix} \begin{bmatrix}\underline{U}_2\\ \underline{I}_2'\end{bmatrix}$$

$$\underline{I}_2' := -\underline{I}_2$$

$$\underline{a}_{11}=\frac{\underline{U}_1}{\underline{U}_2}\Big|_{\underline{I}_2'=0} \qquad \underline{a}_{12}=\frac{\underline{U}_1}{\underline{I}_2'}\Big|_{\underline{U}_2=0} \qquad \underline{a}_{21}=\frac{\underline{I}_1}{\underline{U}_2}\Big|_{\underline{I}_2'=0} \qquad \underline{a}_{22}=\frac{\underline{I}_1}{\underline{I}_2'}\Big|_{\underline{U}_2=0}$$

Hybridmatrix $\underline{H}$

$$\begin{bmatrix}\underline{U}_1\\ \underline{I}_2\end{bmatrix} = \begin{bmatrix}\underline{h}_{11} & \underline{h}_{12}\\ \underline{h}_{21} & \underline{h}_{22}\end{bmatrix} \begin{bmatrix}\underline{I}_1\\ \underline{U}_2\end{bmatrix}$$

$$h_{11}=\frac{\underline{U}_1}{\underline{I}_1}\Big|_{\underline{U}_2=0} \qquad \underline{a}_{12}=\frac{\underline{U}_1}{\underline{U}_2}\Big|_{\underline{I}_1=0} \qquad h_{21}=\frac{\underline{I}_2}{\underline{I}_1}\Big|_{\underline{U}_2=0} \qquad \underline{a}_{22}=\frac{\underline{I}_2}{\underline{U}_2}\Big|_{\underline{I}_1=0}$$

ZT ist reziprok $\Leftrightarrow$ $\underline{z}_{12}=\underline{z}_{21}$ — symetrisch $\Leftrightarrow$ $\underline{z}_{12}=\underline{z}_{21}$ $\underline{z}_{11}=\underline{z}_{22}$

$\Leftrightarrow$ $\underline{y}_{12}=\underline{y}_{21}$ — $\Leftrightarrow$ $\underline{y}_{12}=\underline{y}_{21}$ $\underline{y}_{11}=\underline{y}_{22}$

$\Leftrightarrow$ $\det A = 1$

$\Leftrightarrow$ $\underline{h}_{12}=-\underline{h}_{21}$

$\Leftarrow$ reines RLCÜ – ZT

Zweitorverbindungen

Reihenverbindung	Kettenschaltung	Parallelverbindung
$\underline{Z} = \underline{Z}_1+\underline{Z}_2$	$\underline{A} = \underline{A}_1\underline{A}_2$	$\underline{Y} = \underline{Y}_1+\underline{Y}_2$

Differentialgleichungen

DGL für Netzwerke mit einem Energiespeicher bei sprungförmiger Erregung

$y(t) = A + B\,e^{-t/T}$ mit $T = \frac{L}{R} = RC$

$A = y(\infty)$ $B = y(0) - A$

Laplace-Transformation

$$F(p) := \int_0^\infty f(t)\,e^{-pt}\,dt$$

$$f(t) := \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(p)\,e^{pt}\,dp \qquad (\sigma = const > \sigma_{min},\ dp = jd\omega)$$

Eigenschaften

$c_1 f_1(t) + c_2 f_2(t) \quad \circ\!-\!\bullet \quad c_1 F_1(p) + c_2 F_2(p)$

$s(at)\,f(at) \quad \circ\!-\!\bullet \quad \frac{1}{a}\,F(\frac{p}{a})$

$s(t-t_0)\,f(t-t_0) \quad \circ\!-\!\bullet \quad e^{-pt_0}\,F(p)$

$e^{at}\,s(t)\,f(t) \quad \circ\!-\!\bullet \quad F(p-a)$

$t^n\,s(t)\,f(t) \quad \circ\!-\!\bullet \quad (-1)^n\,(\frac{d}{dp})^n\,F(p)$

$s(t)\,f^{(n)}(t) \quad \circ\!-\!\bullet \quad p^n\,F(p) - \sum_{\upsilon=0}^{n-1} f^{(\upsilon)}(0+)\,p^{n-1-\nu}$

Übertragungsfunktion und Einschwingvorgang

$Y(p) = H(p)\,X(p)$

$$y(t) = \int_{0-}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

Sprungantwort a(t)

$a(\infty) = H(0)$ $a(0+) = H(\infty)$

$$a(t) = \int_{0-}^{t} h(\tau)\,d\tau$$

Impulsantwort h(t)

$h(t) \quad \circ\!-\!\bullet \quad H(p)$ $h(t) = \frac{da(t)}{dt}$

Nichtlineare Netzwerke

Nichtlineare Induktivität

Stromgesteuert: $\Phi = f_L(i,t)$ $\quad u = \frac{d\Phi}{dt} = \frac{d\, f_L(i,t)}{di}\frac{di}{dt} + \frac{f_L(i,t)}{dt}$

Flußgesteuert: $i = g_L(\Phi,t)$

lin. Zeitvariant: $\Phi = L(t)\, i(t)$ $\quad u = \frac{d\Phi}{dt} = L(t)\frac{di(t)}{dt} + \frac{dL(t)}{dt}\, i(t)$

Energie: $W_L = \int_{\Phi(t_1)}^{\Phi(t_2)} g_L(\Phi)\, d\Phi$

Nichtlineare Kapazität

Spannungsgesteuert: $q = f_C(u,t)$ $\quad i = \frac{dq}{dt} = \frac{d\, f_C(u,t)}{du}\frac{du}{dt} + \frac{f_C(u,t)}{dt}$

Ladungsgesteuert: $u = g_C(q,t)$

lin. Zeitvariant: $q = C(t)\, i(t)$ $\quad i = \frac{dq}{dt} = C(t)\frac{du(t)}{dt} + \frac{dC(t)}{dt}\, u(t)$

Energie: $W_C = \int_{q(t_1)}^{q(t_2)} g_C(q)\, dq$

Lösung der DGL für NL-Netzwerke erster Ordnung bei Polygonzugapproximation

$$\frac{dy}{dt} = c_1 y + c_0 \qquad c_1,\ c_0 = \text{const.}$$

$$y(t) = -\frac{c_0}{c_1} + \left[y(t_0) + \frac{c_0}{c_1}\right] e^{c_1(t-t_0)}$$

Systemtheorie

Umwandlung von Differentialgleichungen in Zustandsgleichungen

$$\frac{d^q y}{dt^q} + \alpha_{q-1}\frac{d^{q-1}y}{dt^{q-1}} + \ldots + \alpha_1\frac{dy}{dt} + \alpha_0 y = \beta_{q-1}\frac{d^{q-1}x}{dt^{q-1}} + \ldots + \beta_1\frac{dx}{dt} + \beta_0$$

falls ein Term $\beta_q \frac{d^q x}{dt^q}$ auftritt, Substitution $\mathit{y} = y - \beta_q x$ anwenden

Erstes Verfahren

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & \ldots & -\alpha_{q-1} \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \ldots \\ 1 \end{bmatrix} x \qquad y = \begin{bmatrix}\beta_0 & \beta_1 & \beta_2 & \ldots & \beta_{q-1}\end{bmatrix}\mathbf{z}$$

• System in dieser Darstellung ist stets steuerbar (Steuerungsnormalform)

Zweites Verfahren

$$\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_{q-1} \end{bmatrix} \mathbf{z} + \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \alpha_{q-1} & 1 & 0 & \dots & 0 \\ \alpha_{q-2} & \alpha_{q-1} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_1 & \alpha_2 & \dots & \alpha_{q-1} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \beta_{q-1} \\ \beta_{q-2} \\ \beta_{q-3} \\ \dots \\ \beta_0 \end{bmatrix} x$$

$$y = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix} \mathbf{z}$$

• System in dieser Darstellung ist stets beobachtbar (Beobachtungsnormalform)

Lösung der Zustandsgleichungen

$$\mathbf{z}(t) = \phi(t-t_0)\mathbf{z}(t_0) + \int_{t_0}^{t} \phi(t-\sigma)\mathbf{B}\mathbf{x}(\sigma)\, d\sigma \qquad \text{mit } \phi(t) = e^{\mathbf{A}t}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{z}(t) + \mathbf{D}\mathbf{x}(t)$$

Berechnung von $\phi(t)$ über Cayley-Hamilton-Theorem

$$e^{\mathbf{A}t} = \alpha_0(t)\,\mathbf{E} + \alpha_1(t)\,\mathbf{A} + \,..... + \alpha_{q-1}(t)\,\mathbf{A}^{q-1}$$

$$e^{p_\mu t} = \alpha_0(t) + \alpha_1(t)\,p_\mu + \,..... + \alpha_{q-1}(t)\,p_\mu^{\,q-1}$$ (für einfache Eigenwerte)

$$\frac{d^\nu}{dp_\mu^{\,\upsilon}}\left[e^{p_\mu t} = \alpha_0(t) + \alpha_1(t)\,p_\mu + \,..... + \alpha_{q-1}(t)\,p_\mu^{\,q-1}\right] \quad \upsilon = 0,1,...,r_\mu - 1$$ (für mehrfache Eigenwerte)

Berechnung von $\phi(t)$ über Diagonalisierung von A

$$\phi(t) = e^{\mathbf{A}t} = \mathbf{M} \begin{bmatrix} e^{p_1 t} & 0 & 0 & 0 \\ 0 & e^{p_2 t} & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & e^{p_{q1} t} \end{bmatrix} \mathbf{M}^{-1}$$ (im Fall einfacher Eigenwerte)

für $$\mathbf{A} = \mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_{q-1} \end{bmatrix} \quad \text{ist } \mathbf{M} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ p_1 & p_2 & \dots & p_q \\ \dots & \dots & \dots & \dots \\ p_{1_{q-1}} & p_{2_{q-1}} & \dots & p_{q_{q-1}} \end{bmatrix}$$ (einfache EW´te)

$$\det(p\mathbf{E} - \mathbf{F}) = \sum_{\upsilon=0}^{q-1} \alpha_\nu p^\nu + p^q$$

Berechnung von $\phi(t)$ über Laplace-Transformation

$$\phi(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(p\mathbf{E}-\mathbf{A})^{-1}\right]$$

$$\mathbf{Z}(p) = (p\mathbf{E}-\mathbf{A})^{-1}\mathbf{z}(0+) + (p\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}\mathbf{X}(p)$$

$$\mathbf{Y}(p) = \mathbf{C}(p\mathbf{E}-\mathbf{A})^{-1}\mathbf{z}(0+) + \left[\mathbf{C}(p\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}\right]\mathbf{X}(p)$$